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NONTRIVIAL COBORDISMS WITH GEOMETRICALLY FINITE HYPERBOLIC STRUCTURES

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Abstract

This paper establishes a new four-dimensional phenomenon: there exist nontrivial (but homologically trivial) four-dimensional cobordisms which are hyperbolic manifolds with geometrically finite structure, i.e. those obtained by identifying the sides of a finite-sided convex polyhedron in the hyperbolic space H^n. In the three-dimensional case analogous cobordisms are trivial: they coincide with the product $S_g \times [0,1]$. The present construction is based on the investigation of geometrically finite Kleinian groups in space, and on the construction of the above groups with a wild sphere as the limit set.

1. Formulation of the problem

It is well known (Marden [11]) that a three-dimensional manifold $M(G) = (H^3 \cup O(G))/G$ uniformized by a geometrically finite Kleinian group G with an invariant contractible component O_0 of the discontinuity set $O(G)$ is organized as follows:

(i) if the manifold $M(G)$ is compact, then it is a surface layer, i.e., the product $S_0 \times [0,1]$ where a surface $S_0 = O_0/G$;

(ii) if $M(G)$ is noncompact, then it is obtained from the surface layer $S_0 \times [0,1]$ by attaching a finite number of collars homeomorphic to $S^1 \times [0,1] \times [0,1)$. In this case the surface $S_0 = O_0/G$ may be obtained from a compact surface by a finite number of punctures.

A question arises: To what extent holds the analogy with the surface layer for the manifold $M(G)$ in higher dimensions, at least in compact case?

We can consider analogies of the $(n+1)$-dimensional layer $(n \geq 3)$ with various degrees of generality:

(a) the product of an n-manifold $M_0 = O_0/G$ by the segment;

(b) a manifold M whose boundary $\operatorname{bd} M$ consists of two components N_0 and N_1 and such that the triple $(M; N_0, N_1)$ is an h-cobordism;

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(c) the manifold M with the boundary components N_0 and N_1 which is a homologically trivial cobordism:

$$H_*(M, N_0) = H_*(M, N_1) = 0.$$

In all these cases the answer to the above question may be as follows: the manifold $M(G) = (H^{n+1} \cup O(G))/G$ is a "surface layer"

(1) in the sense of (a) if the group $G \subset \text{Möb}_n$ is a quasiconformal conjugation of some Fuchsian group in $\overline{R}^n = \text{bd}\, H^{n+1}$;

(2) in the sense of (b) if the group $G \subset \text{Möb}_n$ has two invariant contractible components $O_0, O_1 \subset (G) \subset \overline{R}^n$ (Theorem 3.4);

(3) in the sense of (c) if the group $G \subset \text{Möb}_n$ has an invariant contractible component $O_0 \subset O(G)$ (Theorem 3.2 and Corollary 3.3).

Moreover, and this is the main result of the present paper (Theorem 5.1), there exist four-dimensional manifolds $M(G)$ (whose interior H^4/G has geometrically finite hyperbolic structure) which are homologically trivial cobordisms (realizing (3)) but without the properties of h-cobordism, i.e. not satisfying (b).

To prove this, in §4 we construct a geometrically finite Kleinian group G in $\overline{R}^3$ whose limit set $L(G)$ is a sphere wildly imbedded into R^3 sphere which divides the discontinuity set $O(G)$ into two invariant components O_0 and O_1, one of them being contractible.

Note the following question which is a special case of S. P. Novikov's conjecture on h-cobordisms of the type $K(\pi, 1)$, and still is open (see also Remark 5.3):

Is the h-cobordism $(M(G); O_0/G, O_1/G)$ trivial if it corresponds to case (2), i.e. to the group $G \subset \text{Möb}_n$ with two G-invariant contractible components $O_0, O_1 \subset O(G)$?

We would like to thank O. Ya. Viro for a helpful conversation concerning the present work.

2. Preliminaries

Let Möb_n be the group of all Möbius transformations (preserving orientation) in the space $\overline{R}^n = R^n \cup \{\infty\}$, and let G be its Kleinian subgroup, i.e. the discrete group whose limit set $L(G)$ does not coincide with $\overline{R}^n$ (the discontinuity set $O(G) = \overline{R}^n - L(G) \neq \varnothing$). The group Möb_n acts isometrically in the hyperbolic $(n+1)$-space H^{n+1} which is $R_+^{n+1} = (x \in R^{n+1} : x_{n+1} > 0)$ with the metric $ds^2 = |dx|^2/x_{n+1}^2$.

A fundamental polyhedron $P \subset H^{n+1}$ of a discrete group $G \subset \text{Möb}_n$ is a polyhedron whose images $G(P)$ yield a locally finite covering of H^{n+1} such that $g(\operatorname{int} P) \cap \operatorname{int} P = \varnothing$ for every $g \in G$, $g \neq \mathrm{id}$. A group $G \subset \text{Möb}_n$ is geometrically finite iff a finite-sided fundamental polyhedron $P \subset H^{n+1}$ exists for it.

The determining properties of geometrically finite Möbius groups may be formulated as follows (for $n \geq 3$ see [4], [5]):

Theorem 2.1. *For a discrete torsion-free group $G \subset \text{Möb}_n$ the following properties are equivalent:*

(1) *G is geometrically finite;*

(2) *the limit set $L(G)$ consists of approximation points and parabolic cusps;*

(3) *for some (any) $r > 0$ the r-neighborhood $U_r(M_G) \subset M(G)$ of the minimal convex retract $M_G \subset H^{n+1}/G$ of the manifold $M(G)$ has finite volume;*

(4) *the submanifold $(M_G)_{[r,\infty)}$ obtained from M_G by cutting off its r-thin parts is compact.*

Note that the above-mentioned minimal convex retract M_G of the manifold $M(G)$ may be characterized as the minimal convex submanifold M_G of the hyperbolic manifold $H^{n+1}/G = \operatorname{int} M(G)$ for which the imbedding $M_G \subset M(G)$ induces the isomorphism of fundamental groups.

An isomorphism $i\colon G \to G'$ of two discrete Möbius groups G and G' is said to be type-preserving if it carries parabolic elements of G bijectively onto parabolic elements of G'. If $A, A' \subset \overline{R}^n \cup H^{n+1}$ are some invariant sets corresponding to groups G and G', we say that a map $f\colon A \to A'$ induces i if $f(g(x)) = i(g)(f(x))$ for every $g \in G$ and $x \in A$; we say also that f is G-compatible (or, if $G = G'$ and $i = \mathrm{id}$, f is said to be a G-equivariant map).

We formulate the properties of isomorphisms of geometrically finite groups in $\overline{R}^n$ which are necessary below in the following statement, which is a partial case of more general statements of P. Tukia (see [15, Theorem 3.3 and Lemma 3.7]):

Theorem 2.2. *Let G and G' be geometrically finite Möbius groups in $\overline{R}^n$ and let $i\colon G \to G'$ be a type-preserving isomorphism. Then:*

(1) *there is a homeomorphism $f_i\colon L(G) \to L(G')$ of the limit sets (the unique one if G is a nonelementary group), inducing the isomorphism i;*

(2) *if $A \subset O(G)$ is a G-invariant set with the compact factor A/G and if $f\colon A \to O(G')$ is a continuous map inducing i, then f and the map f_i define together a continuous map $\hat{f}\colon L(G) \cup A \to \overline{R}^n$ which is an imbedding if f is.*

Let M be some compact $(n+1)$-dimensional manifold whose boundary $\operatorname{bd} M$ consists of two disjoint connected closed n-manifolds N_0 and N_1, $N_0 \cap N_1 = \varnothing$. Then the triple $(M; N_0, N_1)$ is called a homologically trivial cobordism if all the relative homology groups are trivial:

$$H_*(M, N_0) = H_*(M, N_1) = 0. \tag{2.1}$$

The triple $(M; N_0, N_1)$ of compact manifolds with boundaries is called a homologically trivial cobordism with boundary if $N_0, N_1 \subset \operatorname{bd} M$, $N_0 \cap N_1 = \varnothing$, and for the boundary $dM = \operatorname{bd} M - (N_0 \cup N_1)$ the equality

$$H_*(M, N_0) = H_*(M, N_1) = H_*(dM, \operatorname{bd} N_0) = H_*(dM, \operatorname{bd} N_1) = 0 \tag{2.2}$$

is valid. (If in these definitions equalities (2.1) and (2.2) are replaced by the requirement of triviality of relative homotopic groups, then the triple $(M; N_0, N_1)$ is said to be an h-cobordism or h-cobordism with boundary.)

3. Invariant components of Kleinian groups and cobordisms

It is well known that geometrically finite nonelementary Kleinian groups on the plane whose discontinuity set $O(G)$ contains a contractible G-invariant component O_0 may be one of the following two kinds (see [1], [2]):

They are either quasi-Fuchsian groups whose discontinuity set consists of two invariant contractible components, or nondegenerate B-groups whose discontinuity set, besides the above component O_0, contains an infinite number of components O_i. All these additional components are noninvariant, but form a finite number of classes of G-equivalent components.

In both cases the three-manifold $M(G) = (H^3 \cup O(G))/G$ uniformized by such groups has the following structure (see Marden [11]):

In the former case $M(G)$ is homeomorphic to the product of the surface $N_0 = O_0/G$ by the closed segment $I = [0,1]$. In the latter case the manifold $M(G)$ also, in a certain sense, looks like the product N_0 by I. Namely, there exists the compactification $\hat{M}$ of the manifold $M(G)$ which is homeomorphic to the product $\hat{N}_0$ by I (where $\hat{N}_0$ is the compactification of the surface $N_0 = O_0/G$ preserving the fundamental group $\pi_1(N_0) = \pi_1(\hat{N}_0)$) and the difference $\hat{M} - M(G)$ is the union of the finite number of cylinders $S^1 \times I$.

As expected, for large $n \geq 3$ the situation proves to be more complicated. This is shown by examples constructed by A. V. Tetenov (see [12], [10]) of infinitely generated Kleinian groups in $\overline{R}^n$, $n \geq 3$, whose discontinuity set $O(G)$ can consist of any number of invariant components, even simply connected ones. However, despite these examples the analogy with the two-dimensional case (for geometrically finite groups) is strong enough. Namely, the following statements (for proofs, see [13], [14]) are valid.

Theorem 3.1. *Let G be a geometrically finite nonelementary Kleinian group in $\overline{R}^n$, $n \geq 2$, with a contractible invariant component O_0 of the discontinuity set $O(G)$. Then $O(G)$ consists of either two invariant components O_0 and O_1 or O_0 and an infinite number of noninvariant components O_i.*

Theorem 3.2. *Let G be a geometrically finite Kleinian torsion free group in $\overline{R}^n$, $n \geq 2$, having invariant contractible component $O_0 \subset O(G)$, and let $N_0 = O_0/G$. Then in the manifold $M(G) = (H^{n+1} \cup O(G))/G$ there exists a compact $(n+1)$-dimensional submanifold M' with the following properties:*

(i) *M is obtained from M' by attaching an open collar $dM' \times [0,1)$ to the boundary $dM' = \mathrm{bd}\, M' - \mathrm{bd}\, M$ of the submanifold M' in M;*

(ii) *connected components of the collar $dM' \times [0,1)$ are homeomorphic to the cylinders*

$$T^{n-k} \times B^k \times [0,1), \qquad 1 \leq k \leq n-1$$

(*here B^k is a closed k-dimensional ball, $T^{n-k} = S^1 \times \cdots \times S^1$*);

(iii) *the boundary $\mathrm{bd}\, M$ contains connected disjoint n-dimensional manifolds with boundary N_0' and N_1', such that*

$$\pi_*(M', N_0') = 0 \quad \text{and} \quad H_*(M', N_1') = 0$$

and the cobordism with boundary $(M'; N_0', N_1')$ is homologically trivial. In this case,

$$N_0' = N_0 \cap M', \qquad N_1' \supset M' \cap (\mathrm{bd}\, M - N_0),$$
$$\mathrm{bd}\, N_0' \approx \mathrm{bd}\, N_1', \qquad \mathrm{bd}\, M' = N_0' \cup N_1' \cup (\mathrm{bd}\, N_0' \times [0,1]).$$

Directly from this fact and from Theorem 2.1 we obtain

Corollary 3.3. *Let a Kleinian group G from Theorem 3.2 have no parabolic elements. Then the compact manifold $M(G)$ has two boundary components $N_0 = O_0/G$ and $N_1 = (O(G) - O_0)/G$, and the triple $(M(G); N_0, N_1)$ is a homologically trivial cobordism.*

This result may be strengthened if we neglect the condition of geometric finiteness of the group G:

Theorem 3.4. *Let G be a Kleinian group in $\overline{R}^n$, $n \geq 2$, having two invariant contractible components $O_0, O_1 \subset O(G)$ with compact factor-manifolds $N_0 = O_0/G$ and $N_1 = O_1/G$. Then the manifold $M(G)$ is also compact, the group G is geometrically finite, $O(G) = O_0 \cup O_1$, and the triple $(M(G); N_0, N_1)$ is an h-cobordism.*

We shall briefly outline a direct proof of Corollary 3.3, since it is essential for the proof of our main result in §5.

Proof of Corollary 3.3. The group G has no parabolic elements; therefore, by Theorem 2.1, the minimal convex retract M_G of the manifold $M(G)$ (and hence, the manifold $M(G)$) is compact.

The manifold $M(G)$ and the component $N_0 = O_0/G$ of its boundary are both the spaces of type $K(G,1)$. The inclusion $N_0 \subset M(G)$ induces the isomorphism of the fundamental group

$$\pi_1(N_0) \to \pi_1(M(G)),$$

and thus it is a homotopy equivalence, which implies

$$H_*(M, N_0) = 0. \tag{3.1}$$

Then, using Poincare duality, we obtain that

$$H_*(M, \operatorname{bd} M - N_0) = 0, \tag{3.2}$$

too.

Property (3.2) implies that $H_0(\operatorname{bd} M - N_0) = Z$, i.e. $\operatorname{bd} M - N_0$ consists of only one component N_1, $N_1 = (O(G) - O_0)/G$, where

$$H_*(M, N_1) = 0. \tag{3.3}$$

By (3.1) and (3.3) the proof is complete.

4. Wild spheres as the limit sets of geometrically finite groups

We base our construction of geometrically finite Kleinian groups $G \subset$ Möb$_n$, whose limit set $L(G)$ is a wild sphere, on an idea of periodicity of knotting used by the first author for the construction of the wildly knotted curve $L(G)$ [3], [10].

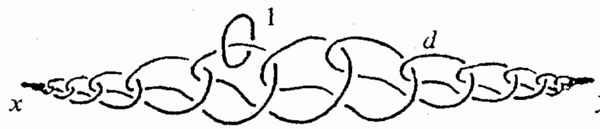

FIGURE 1

Let us consider the Fox-Artin arc $d \subset \overline{R}^3$ (knotted periodically; see [8]) with endpoints x and y (see Figure 1). By "periodically" we mean that d is invariant for the action of some cyclic group, generated by a hyperbolic transformation $h \in$ Möb$_3$, such that $h(x) = x$ and $h(y) = y$. Moreover, if $I(h) = (x\colon |Dh(x)| = 1)$ and $I(h^{-1})$ are the isometric spheres of h, $h(\operatorname{ext} I(h)) = \operatorname{int} I(h^{-1})$, then $I(h) \cap d = (x_1, x_2, x_3)$ and $I(h^{-1}) \cap d = (x_1', x_2', x_3')$ where $h(x_i) = x_i'$ and these points x_i and x_i' are placed on d in the following order:

$$x_1, x_2, x_3, x_1', x_2', x_3'.$$

The intersection d_h of the arc d and $\operatorname{ext} I(h) \cap \operatorname{ext} I(h^{-1})$ consists of three arcs (x_1, x_2), (x_3, x_1'), (x_2', x_3') and forms the period of d, shown in Figures 2 and 3.

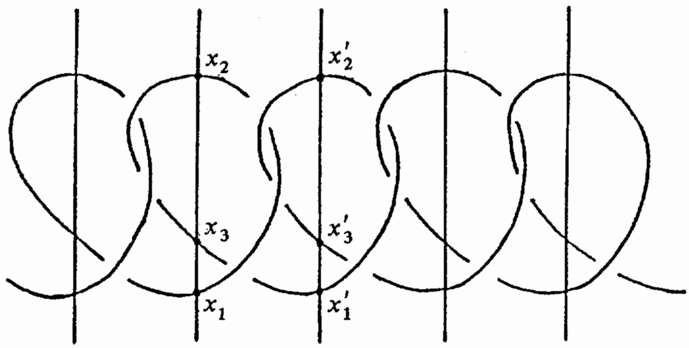

FIGURE 2

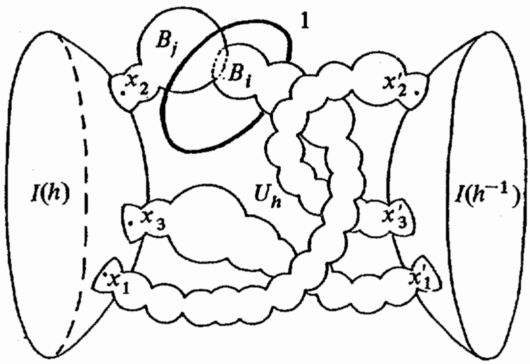

FIGURE 3

Now we take a neighborhood U_h of the three arcs of d_h in $\operatorname{ext} I(h) \cap \operatorname{ext} I(h^{-1})$ consisting of three disjoint tubes shown in Figure 3. For our further needs we can form this neighborhood of a finite number of consequently overlapping balls B_i, in accordance with the established periodicity of d, manifested here by the fact that if $x_k \in B_i \cap I(h)$ and $x'_k \in B_j \cap I(h^{-1})$ then $h(B_i \cap I(h)) = B_j \cap I(h^{-1})$.

It is easy to see that the closure of the union of spherical annuli

$$X_i = \operatorname{bd} B_i - \left(\bigcup_{j \neq i} B_j \cup \operatorname{int} I(h) \cup \operatorname{int} I(h^{-1}) \right)$$

and their h^m-images, $m \in Z$, is the boundary of the fattening $U(d) = \bigcup(h^m(\overline{U}_h)\colon m \in Z) \cup \{x, y\}$ of the arc d, and is a wild sphere S^* in R^3 (see Figure 4).

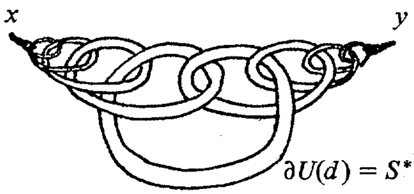

FIGURE 4

Now we can form a finite family C of spheres S_j (contained in some regular neighborhood of the boundary of the three tubes), possessing the following properties:

1. The union of the annuli X_i is covered by interiors of S_j.

2. For each i, j either $S_j \cap B_i = \emptyset$ or S_j is orthogonal to $\operatorname{bd} B_i$; this also holds for $I(h)$ and $I(h^{-1})$ taken instead of B_i.

3. If $S_i \cap S_j \neq \emptyset$ then the dihedral angle between them is π/m, $m \in N$.

4. If $S_j \cap S_k$ is nonempty then there is a common annulus X_i for which $S_j \cap X_i \neq \emptyset$ and $S_k \cap X_i \neq \emptyset$.

5. There is one-to-one correspondence between spheres $S_j \in C$ crossing $I(h)$ and spheres $S'_j \in C$ crossing $I(h^{-1})$ so that $h(S_j) = S'_j$.

In other words, we form a finite "bubble cover" of $\operatorname{bd} U_h$ with good angles between the bubbles and right angles between the bubbles and $\operatorname{bd} B_i$, and respecting the periodicity. One can see easily that the freedom of choice of the balls B_i (so as d and h) permits us to vary moduli of spherical annuli X_i and thus obtain such a family C.

Indeed, taking into account the rigidity of circular coverings of a sphere (which is connected with the rigidity of hyperbolic polyhedra and hyperbolic space forms), we will, besides the above-mentioned arguments of existence, give a construction of such a covering C for the chosen type of a wild knot.

Let us consider a right prism P in R^3 with height 13, whose base is a polygon which is a union of 28 equal regular hexagons with unit sides. Here the centers of the extremal hexagons are the vertices of a regular triangle with side equal to $6\sqrt{3}$ (see Figure 5). Let us enumerate all the hexagons as shown in the picture, so that the three extremal hexagons have the numbers 1, 7 and 28, and central one has the number 16.

Divide the prism P into (28×13) small hexagonal prisms $P(k,n)$ of unit height enumerated by pairs (k,n) where k, $1 \leq k \leq 13$, is the "floor" of the large prism P containing $P(k,n)$ and n, $1 \leq n \leq 28$, is the number of a small hexagon which is a projection of $P(k,n)$ to the base of P.

Now we shall put in correspondence to the three tubes forming the neighborhood U_h of the link $d_h = (x_1, x_2) \cup (x'_2, x'_3) \cup (x_3, x'_1)$ three disjoint domains

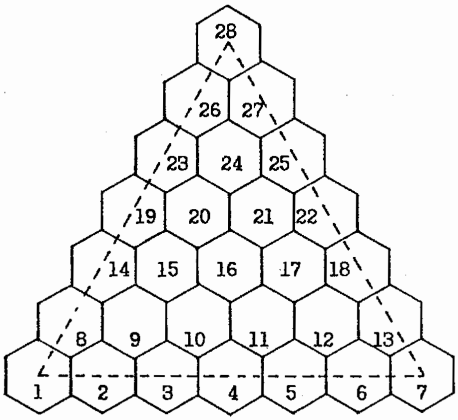

FIGURE 5

$D(x_1, x_2)$, $D(x'_2, x'_3)$ and $D(x_3, x'_1)$, obtained as a union of a number of some small prisms $P(k, n)$ with the numbers from the following sets of pairs:

$(10,14), (10,15), (10,16), (10,17), (10,18)$
$(9,14)$ $(9,18)$
$(8,14)$ $(8,18)$
$(7,14)$ $(7,18)$
$(6,14)$ $(6,18)$
$(5,14)$ $(5,18)$
$(4,14), (4,8), (4,1)$ $(4,18)$
$(3,1)$ $(3,18)$
$(2,1)$ $(2,18)$
$(1,10), (1,3), (1,2), (1,1)$ $(1,18), (1,17)$

$(13,10), (13,3)$ $(13,28), (13,26), (13,23), (13,20)$
$(12,3)$ $(12,28)$
$(11,3)$ $(11,28)$
$(10,3)$ $(10,28), (10,27), (10,25)$
$(9,3)$ $(9,25)$
$(8,3)$ $(8,25)$
$(7,3)$ $(7,25)$
$(6,3)$ $(6,25)$
$(5,3)$ $(5,25)$
$(4,3), (4,10), (4,16), (4,21), (4,25)$

$$\begin{array}{r} (13,17),(13,18),(13,13),(13,7) \\ (12,7) \\ (11,7) \\ (10,7) \\ (9,7) \\ (8,7) \end{array}$$

$$\begin{array}{l} (7,23),(7,20),(7,16),(7,11),(7,5),(7,6),(7,7) \\ (6,23) \\ (5,23) \\ (4,23) \\ (3,23) \\ (2,23) \\ (1,23),(1,20) \end{array}$$

It is essential to remark that we distinguish three square sides on each of the two prisms $P(1,16)$ and $P(13,16)$ which are connected by the domains $D(x_i, x_j)$ constructed above.

Now let S_i be the spheres of radii $\sqrt{3}/3$ with the centers in vertices of prisms $P(k,n)$ forming the domains $D(x_i, x_j)$. If such spheres S_i and S_j intersect, then their centers are the adjacent vertices of some prism $P(k,n)$ and their angle of intersection if $\pi/3$.

Denote by $B(k,n)$ the ball with the center in the center of the prism $P(k,n)$ and of radius $\sqrt{11/12}$. Its boundary sphere $S(k,n)$ is orthogonal to each of the spheres S_i whose centers are the vertices of $P(k,n)$. After that we may regard the balls B_i whose union is the three components of U_h as the balls $B(k,n)$ corresponding to prisms $P(k,n)$ from the domains $D(*,*)$. Here the isometric spheres $I(h)$ and $I(h^{-1})$ are the spheres $S(1,16)$ and $S(13,16)$ correspondingly and points x_i and x_i', $h(x_i) = x_i'$, $1 \le i \le 3$, are the points on these spheres which project along the radii to the centers of distinguished sides of prisms $P(1,16)$, $P(13,16)$, i.e. $(x_1, x_2) \subset D(x_1, x_2)$, $(x_2', x_3') \subset D(x_2', x_3')$, $(x_3, x_1') \subset D(x_3, x_1')$.

The interiors of the spheres S_i do not cover the whole boundary $\operatorname{bd} U_h$, i.e. do not cover all the spherical annuli $X_i \subset S(k,n)$. Still uncovered are the hexagonal and quadrangular domains on these annuli, corresponding to sides of prisms $P(k,n)$. Each of these quadrangular domains on $S(k,n)$ we shall cover by the interiors of five spheres, orthogonal to the sphere $S(k,n)$, four of them being also orthogonal to the spheres S_i, having equal radii and crossing each other at the angle $\pi/3$, and the fifth sphere will cross the previous four orthogonally and will not cross the spheres S_i (see Figure 6).

Each hexagonal domain on $S(k,n)$ we shall, in its turn, cover by the interiors of seven spheres, orthogonal to the sphere $S(k,n)$. Six of them will be

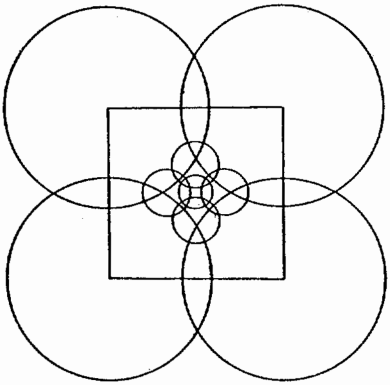

FIGURE 6

of equal radii, orthogonal to the spheres S_i and cross each other at the angle of $\pi/3$; the seventh sphere will cross the six others orthogonally and not cross the spheres S_i (see Figure 7).

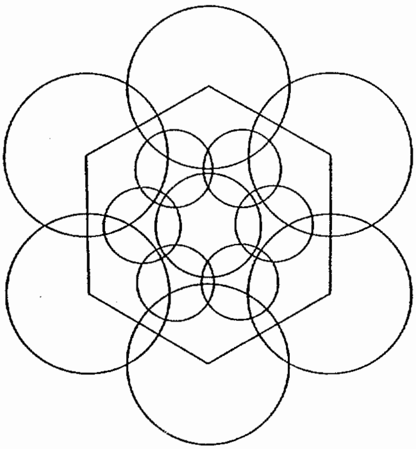

FIGURE 7

The direct computation shows that the obtained covering of the boundary of U_h by interiors of spheres has all the properties of the family C with the only exception that the spheres S_i whose centers are the vertices of disjoint prisms

$$P(1,10), P(1,17), P(1,20) \quad \text{and} \quad P(13,10), P(13,17), P(13,20)$$

cross each other instead of the fact that their interiors cover disjoint spherical annuli $X_i \subset \operatorname{bd} U_h$. Nevertheless, we can subdivide our hexagonal prisms to the finer ones, keeping the prisms $P(1,16)$ and $P(13,16)$ without change. Then the covering, obtained for the corresponding (finer) domains $D(*,*)$ together with the spheres $I(h)$ and $I(h^{-1})$ (almost unchanged) will already possess all of the properties 1–5.

We have to remark here that properties 2 and 4 (2 in the case $S_j \cap B_i \neq \emptyset$ and $S_j \cap B_k \neq \emptyset$ means S_j is orthogonal to $\mathrm{bd}(B_i) \cap \mathrm{bd}(B_k)$) give us the possibility of "bending" of cylinders $\mathrm{bd}\, U_h$ and, hence, of the whole surface S^* along the circles which bound the annuli $h^m(X_i)$, $i \in I$, $m \in Z$, without changing their moduli i, i.e. without changing the dihedral angles between spheres S_j (and their h^m-images).

Let H be a Möbius group generated by the hyperbolic transformation h and by reflections I_j in spheres $S_j \in C$. Property 3 of C leads to discreteness of the group H, while the finiteness of the family C proves its geometrical finiteness. Let F_1 denote the unbounded (in R^3) component of spherical polyhedron

$$\operatorname{ext} I(h) \cap \operatorname{ext} I(h^{-1}) \cap (\operatorname{ext} S_j \colon S_j \in C). \tag{4.1}$$

Let the family C be divided into two subsets:

$$C_1 = (S_j \in C \colon S_j \cap B_i \neq \emptyset \text{ for some } B_i, B_i \cap (x_1, x_2) \neq \emptyset),$$
$$C_0 = C - C_1.$$

Denote by F_0 a spherical polyhedron bounded in R^3, obtained by joining the two bounded components of the polyhedron (4.1), having their sides on spheres of the subfamily C_0, with the h-image of the third bounded component of the polyhedron (4.1) whose sides are placed on spheres of the subfamily C_1. It is clear that F_0 is a connected polyhedron containing the segment (x_3, x_3') of the arc d. Its union with F_1 gives a polyhedron $F = F_0 \cup F_1$ which is a fundamental (unconnected) polyhedron for the group H. As for any group generated by reflections (see [3, Lemma 3.3]), the domains

$$O_0 = \bigcup (g(\overline{F}_0) \colon g \in H) \quad \text{and} \quad O_1 = \bigcup (g(\overline{F}_1) \colon g \in H)$$

are the invariant components for the group H and, since F is a fundamental polyhedron for H, $O(H) = O_0 \cup O_1$.

Let G be the group of finite index in H without elements of finite order and consisting of orientation preserving transformations. For the Kleinian group $G \subset \text{Möb}_3$, clearly:

$$O(G) = O(H) \quad \text{and} \quad L(G) = L(H).$$

Theorem 4.1. *The limit set $L(G)$ of the constructed geometrically finite Kleinian group $G \subset \text{Möb}_3$ is a wild sphere in R^3 dividing the discontinuity set $O(G)$ into two G-invariant components, one of them being a K-quasiconformal ball.*

Proof. For the proof of the theorem it is enough to construct a homeomorphism $\hat{f}: \overline{O}_0 = O_0 \cup L(G) \to \overline{B}$ of the closure of the component O_0 onto a closed ball $\overline{B} \subset R^3$, which is quasiconformal in O_0 and compatible with the group H, and therefore, with the group $G \subset H$.

We shall suppose from now that the family C of spheres is the union of subfamilies C_0 and $h(C_1)$ where C_0 and C_1 were defined above. The (new) family C, thus bounds (together with spheres $I(h)$ and $I(h^{-1})$) a connected polyhedron F_0 which is a fundamental one for the group H in the domain O_0.

Let us take any pair of adjacent balls B_i and B_j from those of which we formed the neighborhood U_h, with $S_{ij} = \operatorname{bd} B_i \cap \operatorname{bd} B_j$. We define a quasiconformal homeomorphism f_{ij} of $B_i \cup B_j$ onto the ball B_i, conformal in a neighborhood of spherical disks $(\operatorname{bd} B_i - \overline{B}_j)$ and $(\operatorname{bd} B_j - \overline{B}_i)$, in the following way. Consider B_i and B_j as a pair of half-spaces whose boundary planes contain the third coordinate axis $(x \in R^3 : x_1 = x_2 = 0)$ and let the dihedral angle between them be w, $0 < w < \pi$. Moreover, regard the plane $(x : x_3 = 0)$ as a complex plane $\mathbf{C} = (z = x_1 + ix_2 : (x_1, x_2) \in R^2)$ and fix a number v such that $0 < v < \pi/2$, $0 < w < \pi - 2v$. Then the quasiconformal homeomorphism f_{ij} is described by its projection on the plane $\mathbf{C} = R^2$ where

$$(4.2) \qquad f_{ij}^{-1}(z) = \begin{cases} z, & |\arg z| \geq \pi - v, \\ z \cdot \exp(iw), & |\arg z| \leq v, \\ z \cdot \exp(iw(1 - (\arg(z) - v)/(\pi - 2v))), & v < \arg z < \pi - v, \\ z \cdot \exp(iw(1 + (\arg(z) + v)/(\pi - 2v))), & v - \pi < \arg z < -v. \end{cases}$$

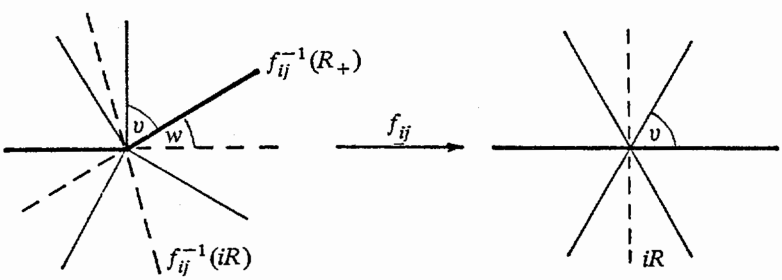

FIGURE 8

Taking the composition of all such quasiconformal homeomorphisms f_{ij} (running over all the neighboring B_i and B_j, finite in their number), we obtain a quasiconformal homeomorphism f_0 of the polyhedron F_0 into a ball B, and sending the sides of F_0 to the sides of some polyhedron $F_0' = f_0(F_0)$ which lay on spheres, orthogonal to the sphere $\mathrm{bd}\, B$ (if $I(h)$ or $I(h^{-1})$ intersect with some B_i, then it is orthogonal $\mathrm{bd}\, B_i$). Since the homeomorphisms f_{ij} are conformal in the neighborhoods of disks $(\mathrm{bd}\, B_i - \overline{B}_j)$ and $(\mathrm{bd}\, B_j - \overline{B}_i)$, we obtain, taking into account properties 2 and 4 of the family C, that all the corresponding dihedral angles W and $f_0(W)$ on the boundaries of F_0 and $f_0(F_0)$ are equal. This proves the following fact:

Let H' be a Möbius group generated by a hyperbolic transformation $h^* = f_0^*(h)$, which maps exterior of the sphere containing $f_0(I(h))$ onto interior of the sphere containing $f_0(I(h^{-1}))$, and by reflections in spheres containing the sides of the polyhedron $f_0(F_0)$. Then H' is a discrete group (see [6]) acting on a ball B with a compact factor B/H', and H' is isomorphic to the group H.

Extending the map f_0 to the images $H(\overline{F}_0)$ of the polyhedron $\overline{F}_0$, we obtain an H-compatible K-quasiconformal map $f\colon O_0 \to B$ which conjugates the groups H and H'.

Now it remains to show that f extends continuously to a homeomorphism $\hat{f}$ of closed domains. We obtain that using Tukia's Theorem 2.2.

To finish the proof, we have to demonstrate that the topology sphere $L(G) = \hat{f}^{-1}(\mathrm{bd}\, B)$ is a wild sphere. For that it suffices to show that the fundamental group $\pi_1(O_1) \neq 0$.

Consider a simple loop 1 in the component $O_1 \subset O(H) = O(G)$ shown in Figures 3 and 1 and suppose that it is contractible in O_1. Then by Dehn's lemma (see, for example, [5, Theorem 8.4]), there is a disk $D \subset O_1$ such that $\mathrm{bd}\, D = 1$. Since D is compact it is covered by a finite number of polyhedra $h_i(\overline{F}_1)$, $h_i \in H$. At the same time the nontriviality of 1 in the complement $\overline{R}^3 - d$ implies that $D \cap d \neq \varnothing$. Therefore, there is an $h_i \in H$ such that $h_i(F_1) \cap d \neq \varnothing$. The obtained contradiction finishes the proof.

Remark 4.2. The set of points $z \in L(G)$, where the sphere $L(G)$ is wildly knotted, is a dense subset of $L(G)$. It follows from the density in the limit set $L(G)$ of the group G of the G-orbit of the points x and y (the endpoints of d, fixed by the hyperbolic transformation $h \in G$): compare [5], Lemma 3.16.

Remark 4.3. Our construction of a quasiconformal homeomorphism in the proof of Theorem 4.1 and Remark 4.2 prove the existence of a quasiconformal embedding f of a ball $B^3 \hookrightarrow R^3$ into R^3 which is extended up to an embedding $D \hookrightarrow R^3$ of no open domain D containing the ball B^3 (see also [7]).

5. The topology of the manifold $M(G)$

Now we state and prove the main result of this paper:

Theorem 5.1. *There exist four-dimensional manifolds* $M(G) = (H^4 \cup O(G))/G$ *(with* $\operatorname{int} M(G)$ *provided by the geometrically finite hyperbolic structure) which are homologically trivial cobordisms, but not h-cobordisms.*

Proof. The proof of the theorem follows from our construction of geometrically finite Kleinian groups $G \subset \text{Möb}_3$ in the previous section and from Corollary 3.3.

Actually, the mentioned group G possesses the following properties:

(1) the manifold $M(G)$ is compact (since the torsion free group G has no parabolic elements);

(2) the boundary $\operatorname{bd} M(G)$ consists of two components $N_0 = O_0/G$ and $N_1 = O_1/G$;

(3) O_0 is a contractible G-invariant component and $O_1 = O(G) - O_0$ is a G-invariant component of $O(G)$ with nontrivial fundamental group $\pi_1(O_1)$.

Therefore, using Corollary 3.3, we obtain that the triple $(M(G); N_0, N_1)$ is a homologically trivial cobordism:

$$H_*(M(G), N_0) = H_*(M(G), N_1) = 0.$$

Indeed, since the component O_1 is not simply-connected and $\pi_1(H^4 \cup O_1) = 0$, the kernel of the homomorphism

$$\pi_1(N_1) \to \pi_1(M(G))$$

induced by the inclusion $N_1 \subset M(G)$ is not zero. This gives the nontriviality of $\pi_2(M(G), N_1)$, and completes the proof.

Remark 5.2. It follows from the construction of the group G that there exists a Fuchsian group $G' \subset \text{Möb}_3$ isomorphic to G such that $M(G') = M^3 \times [0,1]$. This shows that supplementary conditions which may guarantee the homotopical triviality of the cobordism $(M(G); N_0, N_1)$, or moreover its triviality in the usual sense, must have nonalgebraic nature.

Remark 5.3. Moreover, from the isomorphism of $\pi_1(N_0) \cong \pi_1(M)$ to the fundamental group of a closed hyperbolic 3-manifold and from the Farrell-Jones result [9] it follows that the Whitehead group $\operatorname{Wh} G$ is trivial (so as $\operatorname{Wh}_2 G = 0$, $\tilde{K}_0(ZG) = 0$, $K_{-m}(ZG) = 0$ for $m > 0$ and $\operatorname{Wh}_m G \otimes \mathbf{Q} = 0$ for all m).

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